

Fermion Systems in Discrete Space-Time

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Abstract

Fermion systems in discrete space-time are introduced as a model for physics on the Planck scale. We set up a variational principle which describes a non-local interaction of all fermions. This variational principle is symmetric under permutations of the discrete space-time points. We explain how for minimizers of the variational principle, the fermions spontaneously break this permutation symmetry and induce on space-time a discrete causal structure.

It is generally believed that the concept of a space-time continuum (like Minkowski space or a Lorentzian manifold) should be modified for distances as small as the Planck length. We here propose a concise model where we assume that space-time is discrete on the Planck scale. Our notion of “discrete space-time” differs from other discrete approaches (like for example lattice gauge theories or spin foam models) in that we do not assume any structures or relations between the space-time points (like the nearest-neighbor relation on a lattice or a causal network). Instead, we set up a variational principle for an ensemble of quantum mechanical wave functions. The idea is that for minimizers of our variational principle, these wave functions should induce relations between the discrete space-time points, which, in a suitable limit, should go over to the topological and causal structure of a Lorentzian manifold.

The concepts outlined here are worked out in detail in a recent book [1]. Furthermore, in this book the connection to the continuum theory is made precise by introducing the notion of the *continuum limit*, and mathematical methods are developed for analyzing our variational principle in this limit. More specifically, in the continuum limit the fermionic wave functions group to a configuration of Dirac seas; for details see [5]. Analyzing our variational principle in the continuum limit gives concrete results for the effective continuum theory; see [1] and the review article [4].

In this short article we cannot enter the constructions leading to the continuum limit. Instead, we introduce the mathematical framework in the discrete setting (Sections 1 and 2) and discuss it afterwards, working out the underlying physical principles (Section 3). We finally describe the spontaneous symmetry breaking and the appearance of a “discrete causal structure” (Section 4).

1 Fermion Systems in Discrete Space-Time

We let $(H, \langle \cdot | \cdot \rangle)$ be a complex inner product space of signature (N, N) . Thus $\langle \cdot | \cdot \rangle$ is linear in its second and anti-linear in its first argument, and it is symmetric,

$$\overline{\langle \Psi | \Phi \rangle} = \langle \Phi | \Psi \rangle \quad \text{for all } \Psi, \Phi \in H,$$

and non-degenerate,

$$\langle \Psi | \Phi \rangle = 0 \quad \text{for all } \Phi \in H \implies \Psi = 0.$$

In contrast to a scalar product, $\langle \cdot | \cdot \rangle$ is *not* positive. Instead, we can choose an orthogonal basis $(e_i)_{i=1, \dots, 2N}$ of H such that the inner product $\langle e_i | e_i \rangle$ equals $+1$ if $i = 1, \dots, N$ and equals -1 if $i = N + 1, \dots, 2N$.

A *projector* A in H is defined just as in Hilbert spaces as a linear operator which is idempotent and self-adjoint,

$$A^2 = A \quad \text{and} \quad \langle A\Psi | \Phi \rangle = \langle \Psi | A\Phi \rangle \quad \text{for all } \Psi, \Phi \in H.$$

Let M be a finite set. To every point $x \in M$ we associate a projector E_x . We assume that these projectors are orthogonal and complete in the sense that

$$E_x E_y = \delta_{xy} E_x \quad \text{and} \quad \sum_{x \in M} E_x = \mathbf{1}. \quad (1)$$

Furthermore, we assume that the images $E_x(H) \subset H$ of these projectors are non-degenerate subspaces of H , which all have the same signature (n, n) . We refer to (n, n) as the *spin dimension*. The points $x \in M$ are called *discrete space-time points*, and the corresponding projectors E_x are the *space-time projectors*. The structure $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M})$ is called *discrete space-time*. A space-time projector E_x can be used to project vectors of H to the subspace $E_x(H) \subset H$. Using a more graphic notion, we also refer to this projection as the *localization* at the space-time point x .

In order to describe the particles of our system, we introduce one more projector P in H , the so-called *fermionic projector*, which has the additional property that its image $P(H)$ is a *negative definite* subspace of H . The vectors in the image of P have the interpretation as the occupied fermionic states of our system, and thus the rank of P gives the *number of particles* $f := \dim P(H)$.

We call the obtained system $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M}, P)$ a *fermion system in discrete space-time*. Note that our definitions involved only three integer parameters: the spin dimension n , the number of space-time points m , and the number of particles f .

2 A Variational Principle

In order to introduce an interaction of the fermions, we shall now set up a variational principle. To this end, we need to form composite expressions in our projectors $(E_x)_{x \in M}$ and P . It is convenient to use the short notations

$$P(x, y) = E_x P E_y \quad \text{and} \quad \Psi(x) = E_x \Psi. \quad (2)$$

The operator $P(x, y)$ maps $E_y(H) \subset H$ to $E_x(H)$, and it is often useful to regard it as a mapping only between these subspaces,

$$P(x, y) : E_y(H) \rightarrow E_x(H) .$$

Using (1), we can write the vector $P\Psi$ as follows,

$$(P\Psi)(x) = E_x P\Psi = \sum_{y \in M} E_x P E_y \Psi = \sum_{y \in M} (E_x P E_y) (E_y \Psi) ,$$

and thus

$$(P\Psi)(x) = \sum_{y \in M} P(x, y) \Psi(y) . \quad (3)$$

This relation resembles the representation of an operator with an integral kernel, and therefore we call $P(x, y)$ the *discrete kernel* of the fermionic projector. Next we define the *closed chain* A_{xy} by

$$A_{xy} = P(x, y) P(y, x) = E_x P E_y P E_x ; \quad (4)$$

it maps $E_x(H)$ to itself. Let $\lambda_1, \dots, \lambda_{2n}$ be the zeros of the characteristic polynomial of A_{xy} , counted with multiplicities. We define the *spectral weight* $|A_{xy}|$ by

$$|A_{xy}| = \sum_{j=1}^{2n} |\lambda_j| .$$

Similarly, one can take the spectral weight of powers of A_{xy} , and by summing over the space-time points we get positive numbers depending only on the form of the fermionic projector relative to the space-time projectors. Our variational principle is to

$$\text{minimize} \quad \sum_{x, y \in M} |A_{xy}^2| \quad (5)$$

by considering variations of the fermionic projector which satisfy the constraint

$$\sum_{x, y \in M} |A_{xy}|^2 = \text{const} . \quad (6)$$

In the variation we also keep the number of particles f as well as discrete space-time fixed. Using the method of Lagrange multipliers, for every minimizer P there is a real parameter μ such that P is a stationary point of the *action*

$$\mathcal{S}_\mu[P] = \sum_{x, y \in M} \mathcal{L}_\mu[A_{xy}] \quad (7)$$

with the *Lagrangian*

$$\mathcal{L}_\mu[A] = |A|^2 - \mu |A|^2 . \quad (8)$$

This variational principle was first introduced in [1]. In [2] it is analyzed mathematically, and it is shown in particular that minimizers exist:

Theorem 2.1 *The variational principle (5, 6) attains its minimum.*

In [2, Section 3] the variational principle is also illustrated in simple examples.

3 Discussion of the Underlying Physical Principles

We come to the physical discussion. Obviously, our mathematical framework does not refer to an underlying space-time continuum, and our variational principle is set up intrinsically in discrete space-time. In other words, our approach is *background free*. Furthermore, the following physical principles are respected, in a sense we briefly explain.

- The **Pauli Exclusion Principle**: We interpret the vectors in the image of P as the quantum mechanical states of the particles of our system. Thus, choosing a basis $\Psi_1, \dots, \Psi_f \in P(H)$, the Ψ_i can be thought of as the wave functions of the occupied states of the system. Every vector $\Psi \in H$ either lies in the image of P or it does not. Via these two conditions, the fermionic projector encodes for every state Ψ the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers. More technically, we can form the anti-symmetric many-particle wave function

$$\Psi = \Psi_1 \wedge \dots \wedge \Psi_f .$$

Due to the anti-symmetrization, this definition of Ψ is (up to a normalization constant) independent of the choice of the basis Ψ_1, \dots, Ψ_f . In this way, we can associate to every fermionic projector a fermionic many-particle wave function which obeys the Pauli Exclusion Principle. For a detailed discussion we refer to [1, §3.2].

- A **local gauge principle**: Exactly as in Hilbert spaces, a linear operator U in H is called *unitary* if

$$\langle U\Psi | U\Phi \rangle = \langle \Psi | \Phi \rangle \quad \text{for all } \Psi, \Phi \in H.$$

It is a simple observation that a joint unitary transformation of all projectors,

$$E_x \rightarrow U E_x U^{-1}, \quad P \rightarrow U P U^{-1} \quad \text{with } U \text{ unitary} \quad (9)$$

keeps our action (5) as well as the constraint (6) unchanged, because

$$\begin{aligned} P(x, y) &\rightarrow U P(x, y) U^{-1}, & A_{xy} &\rightarrow U A_{xy} U^{-1} \\ \det(A_{xy} - \lambda \mathbb{1}) &\rightarrow \det(U(A_{xy} - \lambda \mathbb{1}) U^{-1}) = \det(A_{xy} - \lambda \mathbb{1}), \end{aligned}$$

and so the λ_j stay the same. Such unitary transformations can be used to vary the fermionic projector. However, since we want to keep discrete space-time fixed, we are only allowed to consider unitary transformations which do not change the space-time projectors,

$$E_x = U E_x U^{-1} \quad \text{for all } x \in M. \quad (10)$$

Then (9) reduces to the transformation of the fermionic projector

$$P \rightarrow U P U^{-1}. \quad (11)$$

The conditions (10) mean that U maps every subspace $E_x(H)$ into itself. Hence U splits into a direct sum of unitary transformations

$$U(x) := U E_x : E_x(H) \rightarrow E_x(H), \quad (12)$$

which act “locally” on the subspaces associated to the individual space-time points.

Unitary transformations of the form (10, 11) can be identified with local gauge transformations. Namely, using the notation (2), such a unitary transformation U acts on a vector $\Psi \in H$ as

$$\Psi(x) \longrightarrow U(x) \Psi(x).$$

This formula coincides with the well-known transformation law of wave functions under local gauge transformations (for more details see [1, §1.5 and §3.1]). We refer to the group of all unitary transformations of the form (10, 11) as the *gauge group*. The above argument shows that our variational principle is *gauge invariant*. Localizing the gauge transformations according to (12), we obtain at any space-time point x the so-called *local gauge group*. The local gauge group is the group of isometries of $E_x(H)$ and can thus be identified with the group $U(n, n)$. Note that in our setting the local gauge group cannot be chosen arbitrarily, but it is completely determined by the spin dimension.

- **The equivalence principle:** At first sight it might seem impossible to speak of the equivalence principle without having the usual space-time continuum. What we mean is the following more general notion. The equivalence principle can be expressed by the invariance of the physical equations under general coordinate transformations. In our setting, it makes no sense to speak of coordinate transformations nor of the diffeomorphism group because we have no topology on the space-time points. But instead, we can take the largest group which can act on the space-time points: the group of all permutations of M . Our variational principle is obviously *invariant under permutations* of M because permuting the space-time points merely corresponds to reordering the summands in (5, 6). Since on a Lorentzian manifold, every diffeomorphism is bijective and can thus be regarded as a permutation of the space-time points, the invariance of our variational principle under permutations can be considered as a generalization of the equivalence principle.

Clearly, the permutation symmetry is not compatible with the topological and causal structure of a Lorentzian manifold. Also, at first sight it might seem problematic that our definitions involve *no locality* and *no causality*. We do not consider these principles as being fundamental. Instead, our concept is that the

minimizer P of our variational principle should spontaneously break the above permutation symmetry and should induce a causal structure on the space-time points. This will be outlined in the next section.

4 Spontaneous Generation of a Discrete Causal Structure

The symmetries of a fermion system in discrete space-time can be described abstractly working with unitary representations of finite groups in indefinite inner product spaces. This abstract framework is developed in [3]. We here state one result and discuss it afterwards. As explained above, discrete space-time $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M})$ as well as our variational principle are symmetric under permutations of the space-time points. However, the fermionic projector might destroy this symmetry. The next definition makes precise what we mean by a permutation symmetry of the whole system.

Def. 4.1 *A fermion system in discrete space-time $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M}, P)$ is called **permutation symmetric** if for every permutation σ of the space-time points M there is a unitary transformation U on H such that*

$$UPU^{-1} = P \quad \text{and} \quad UE_xU^{-1} = E_{\sigma(x)} \quad \text{for all } x \in M.$$

The following theorem is proven in [3].

Theorem 4.2 *Suppose that $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M}, P)$ is a fermion system in discrete space-time of spin dimension (n, n) . If the number of space-time points m is sufficiently large and the number of particles f lies in the range*

$$n < f < m - 1,$$

then the system is not permutation symmetric.

Applied in the physically interesting case $n \ll f \ll m$, where the number of particles is much larger than the spin dimension and much smaller than the number of space-time points, this theorem shows that the fermion system is necessarily less symmetric than discrete space-time and our variational principle. In other words, the fermionic projector spontaneously breaks the permutation symmetry. This result can be understood intuitively as follows. One method for building up a fermion system with permutation symmetry would be to localize one or several particles at each space-time point. But for this we would need at least m particles, in contradiction to our hypothesis $f < m$. Another method would be to work with fermionic states which have permutation symmetry. Such fermions would be “completely delocalized” in the sense that after knowing the wave function at one space-time point, we can recover it at any other space-time point by applying the permutation group. The orthogonality of such permutation symmetric states means that these states must even be orthogonal at each space-time point. Since in addition the states are negative definite, we conclude

that there are at most n such permutation symmetric states, not enough to take into account all $f > n$ particles of our system. Using results of the representation theory of finite groups, it is shown that there is indeed no other method for building up fermion systems with permutation symmetry.

The spontaneous breaking of the permutation symmetry implies that the fermionic projector induces a non-trivial relations between the space-time points. The mathematical structure of our variational principle gives us some insight into the nature of these relations. The basic mechanism becomes clear already in the simplest possible case of spin dimension $(1, 1)$ and the value $\mu = 1/2$ of the Lagrange multiplier in (8). In this case, the closed chain A_{xy} is a 2×2 -matrix; we denote the zeros of its characteristic polynomial by λ_{\pm} . Then the Lagrangian (8) becomes

$$\mathcal{L}[A] = |A^2| - \frac{1}{2} |A|^2 = \left(|\lambda_+|^2 + |\lambda_-|^2 \right) - \frac{1}{2} \left(|\lambda_+| + |\lambda_-| \right)^2,$$

and this can be written as

$$\mathcal{L}[A] = \frac{1}{2} \left(|\lambda_+| - |\lambda_-| \right)^2. \quad (13)$$

Now we have a good intuitive understanding of our variational principle: it tries to achieve that the absolute values of λ_+ and λ_- are equal.

The closed chain A_{xy} is a self-adjoint operator on the indefinite inner product space $(H, \langle \cdot | \cdot \rangle)$. If H were a Hilbert space, we could conclude that A_{xy} is diagonalizable with real eigenvalues. However, this result is not true in indefinite inner product spaces. In general, A_{xy} will not be diagonalizable, and even if it is, its eigenvalues will in general not be real. But at least we know that the characteristic polynomial of A_{xy} is real, and this implies that its zeros λ_{\pm} are either both real, or else they must form a complex conjugate pair (i.e. $\overline{\lambda_+} = \lambda_- \notin \mathbb{R}$). These two cases allow us to introduce a notion of causality.

Def. 4.3 *Two discrete space-time points $x, y \in M$ are called **timelike separated** if the zeros λ_{\pm} of the characteristic polynomial of A_{xy} are both real. Conversely, they are said to be **spacelike separated** if the λ_{\pm} form a complex conjugate pair.*

This definition really reflects the structure of the Lagrangian (13). Namely, if x and y are space-like separated, the zeros λ_{\pm} of the characteristic polynomial of A_{xy} form a complex conjugate pair. Hence $|\lambda_+| = |\overline{\lambda_-}| = |\lambda_-|$, and thus the Lagrangian (13) vanishes identically. Computing first variations of the Lagrangian, one sees that these also vanish, and thus $P(x, y)$ does not enter the Euler-Lagrange equations. This can be seen in analogy to the usual notion of causality in Minkowski space that points with space-like separation cannot influence each other.

According to (13), our variational principle tries to achieve that as many pairs of points (x, y) as possible are space-like separated. On the other hand, the fact that P projects onto a definite subspace of H implies that not all pairs of points can be space-like separated (this is made precise by the lower

bounds of the action in [2, Section 4]). Hence we can say that the mathematical structure of our fermion system ensures that certain space-time points will be timelike separated. On the other hand, the variational principle favors spacelike separation. This gives rise to a spontaneous generation of an interesting causal structure.

References

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